3

FUNDAMENTAL THEOREMS OF PLASTIC ANALYSIS AND DESIGN

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3.1 Introduction

The beams and frames discussed so far have been simple enough for them to be analysed by a "direct" approach. B.M. diagrams were easy to draw and number of possible mechanisms was small, so that the complete behaviour of the structures could be visulised. For complex structure number of possible mechanism increase very rapidly and direct solution becomes very difficult. In this situation it is natural to use approximate methods, however, one should have general principles in order to measure the accuracy of the approx. methods.

In this chapter the basic theorems of plastic theory will be established, which provide powerful techniques for practical computation. For proof of the theorems, principle of virtual work will be employed.

3.2 The equation of VIRTUAL WORK

The principle of virtual work is simple, but some surprising techniques can result from its use. Stated simply

If a body in equilibrium is given
a set of small displacements, then the
work done by the external loads on
the external displacements is equal to
the work done by the internal forces
on the internal displacements.

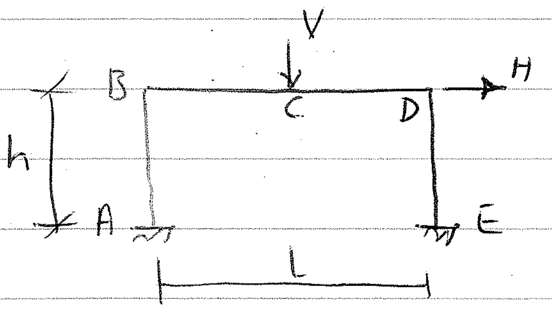
Conditions:

a) The system of displacements must be compatible. However, the deformations need not be "real", i.e the body can be distorted without reference to any loading system. This explains the word "virtual".

b) The internal forces must be in equilibrium with the external loads. However, the internal forces need not be the "actual" forces due to those loads; any equilibrating set of forces may be used in the equation of virtual work.

For the present purpose, only the mechanism type of deformation will be considered. All the internal deformations of the frame will be concentrated at hinges, connected to each other by straight members; The hinge rotations will lead to certain displacements of the joints and loading points.

Suppose we have a portal frame. For certain bending moments M_i $(M_A, M_B, \dots, M_E)$ in equilibrium with W_j (V and H), The equilibrium statement is

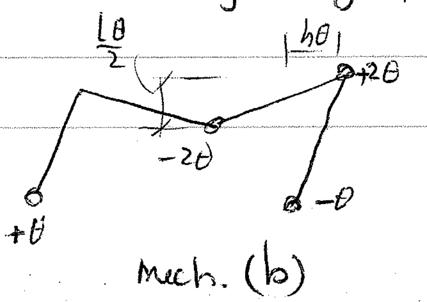


[Loads W_j are in equilibrium with M_i.] (3-1)

or more shortly (W_j, M_i) is an equilibrium set.

M_i is not necessarily the actual bending moments of the frame.

Quite independently of equil., suppose certain hinge rotation ϕ_i lead to joint displacement δ_j, e.g. for portal with the following mechanism we have $(\theta, 0, -2\theta, 2\theta, -\theta)$ at Sections (A, B, C, D, E) leading to $(\frac{L\theta}{2}, h\theta)$ displacements.



Mech. (b)

The deformation statement is then

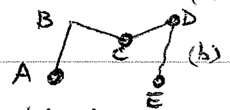
$$\left[\begin{array}{c}\text{Joint displacements } \delta_j \text{ are compatible with hinge} \\ \text{rotations } \phi_i.\end{array}\right] \quad (3\text{-}2)$$

or more shortly: The set (δ_j, ϕ_i) is compatible.

The equation of virtual work combines the above two statements as

$$\sum W_j \delta_j = \sum M_i \phi_i \quad (3\text{-}3)$$

EXAMPLE: For portal frame with imaginary mechanism (b)



(V, H) are in equilibrium with $(M_A, M_B, M_C, M_D, M_E)$ (3-4)

For virtual mechanism (b)

$(\frac{1}{2}l\theta, h\theta)$ are compatible with $(\theta, 0, -2\theta, 2\theta, -\theta)$ (3-5)

Combining (3-4) and (3-5) by means of (3-3)

$$\frac{1}{2}VL + Hh = M_A - 2M_C + 2M_D - M_E \quad (3\text{-}6)$$

This equation holds irrespective of the frame being elastic or plastic, material is not mentioned. What ~~ever~~ the actual values of $M_A, M_B, \ldots, M_E$, should satisfy (3-6).

However, if the virtual mechanism considered (b) is a representation of collapse mechanism, then

$$M_A = M_D = M_p \quad \text{and} \quad M_C = M_E = -M_p \quad (3\text{-}7)$$

Now (3-6) leads to the collapse equation

$$\frac{1}{2}VL + Hh = 6M_p \quad (3\text{-}8)$$

This has been obtained previously by direct work equation.

If M_i in (3-3) correspond to free bending moments, and that a trivial collapse mech. (δ_j, ϕ_i) is under examination, then (3-3) becomes

$$\sum W_j \delta_j = \sum (M_F)_i \phi_i \quad (3\text{-}9)$$

where $(M_F)_i$ now represents a set of free bending moments in equil. with applied load W_j. The collapse equation for the mechanism may be written as

$$\sum W_j \delta_j = \sum (M_p)_i \, |\phi_i| \qquad (3\text{-}10)$$

where $(M_p)_i$ is the numerical value of the full plastic moment at section i. For non-uniform frame $(M_p)_i$ will of course be different for different members. The term $|\phi_i|$ indicates that the numerical values of hinge rotations should be taken, and the sign ignored, since the work dissipated at a plastic hinge is always positive. Thus every term in right-hand side of equation (3-10) is positive.

Combining (3-9) and (3-10)

$$\sum (M_F)_i \, \phi_i = \sum (M_p)_i \, |\phi_i| \qquad (3\text{-}11)$$

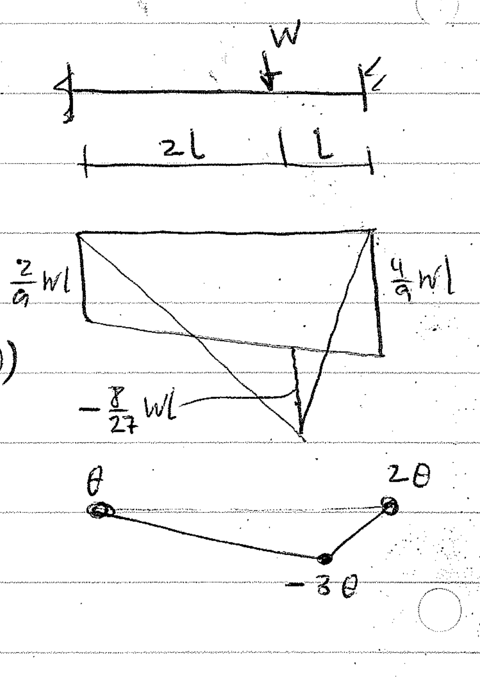
In this equation external loading appears in the form of free B. Ms.

NOTE: It is of interest that the elastic solution, being by definition an equil. set of moments, could be used on the left hand side of (3-11). For example for a beam as shown, from elastic analysis

$$\mathcal{M}_i \equiv \left\{ \tfrac{2}{9}WL, \; -\tfrac{8}{27}WL \;\; \tfrac{4}{9}WL \right\}$$

while the collapse mech. is

$$\phi_i \equiv \{ \theta, \; -3\theta, \; 2\theta \}$$



EQ. (3-11) gives

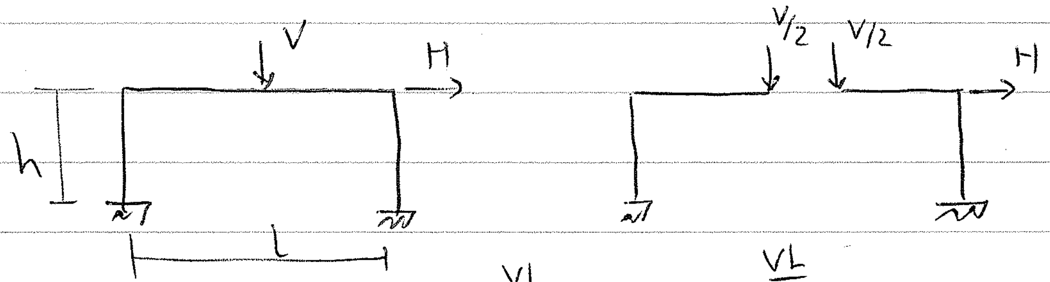
$$\left(\tfrac{2}{9}WL\right)(\theta) + \left(-\tfrac{8}{27}WL\right)(-3\theta) + \left(\tfrac{4}{9}WL\right)(2\theta) = M_p(6\theta)$$

or

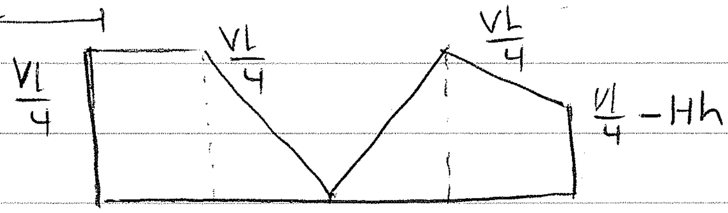
$$M_p = \tfrac{1}{3}WL$$

The result is of course evident from direct inspection.

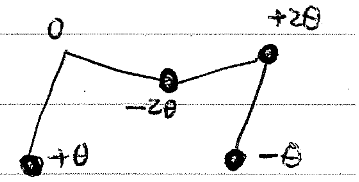
For a portal frame as shown



(a) (b)



(c) Free B.M.D.



(d) mech. (b) type

we have

$$(M_F)_i = \left\{ \frac{VL}{4}, \frac{VL}{4}, 0, \frac{VL}{4}, \frac{VL}{4} - Hh \right\} \qquad (3\text{-}12)$$

for a mechanism mode (b)

$$(\phi_i) = \{ \theta, 0, -2\theta, 2\theta, -\theta \} \qquad (3\text{-}13)$$

Thus the collapse EQ. from (3-11) is

$$\left(\frac{VL}{4}\right)(\theta) + \left(\frac{VL}{4}\right)(0) + (0)(-2\theta) + \left(\frac{VL}{4}\right)(2\theta) + \left(\frac{VL}{4} - Hh\right)(-\theta) = M_p(6\theta) \qquad (3\text{-}14)$$

or

$$\frac{VL}{2} + Hh = 6M_p \qquad (3\text{-}15)$$

as before

Thus the equation of V.W. may be used to obtain direct plastic solutions from a Table of free bending moments.

Further application:

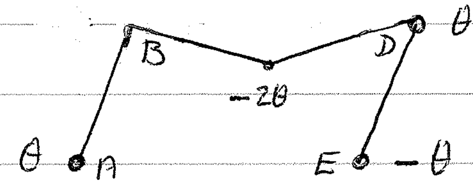
TO check the yield condition, for some mechanism we had to find unknown moments. V.W. EQ. can be used efficiently.

EXAMPLE:

consider the sway mode of portal frame as shown

The collapse bending moments for this mode is

$$(M_c)_i = \{M_p, M_B, -M_p, M_p, -M_p\} \quad (3\text{-}16)$$



We want to check M_B to see whether it is less than $|M_p|$.
The V.W. EQ. can now be used

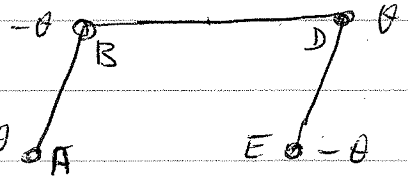
$$\sum (M_c)_i \psi_i = \sum (M_F)_i \psi_i \qquad (3\text{-}17)$$

$(M_c)_i$ and $(M_F)_i$ are two sets of B.Ms in equil. with the same external load. $(M_c)_i$ happens to represent the collapse state, and $(M_F)_i$ a statically determinate state.

The hinge rotation (ψ_i) are any proper mechanism. Thus ψ_i can be taken from e.g. a sway mech.
Auxiliary (assisting)

i.e.

$$(\psi_i) = \{\theta, -\theta, 0, \theta, -\theta\} \qquad (3\text{-}18)$$



Combining in (3-17) yields

$$(M_p)(\theta) + (M_B)(-\theta) + (M_p)(\theta) + (-M_p)(-\theta)$$
$$= (\tfrac{1}{4}Vl)(\theta) + (\tfrac{1}{4}Vl)(-\theta) + (\tfrac{1}{4}Vl)(\theta) + (\tfrac{1}{4}Vl - Hh)(-\theta)$$

Hence

$$M_B = 3M_p - Hh \qquad (3\text{-}19)$$

Substiting M_p from (3-15) we have

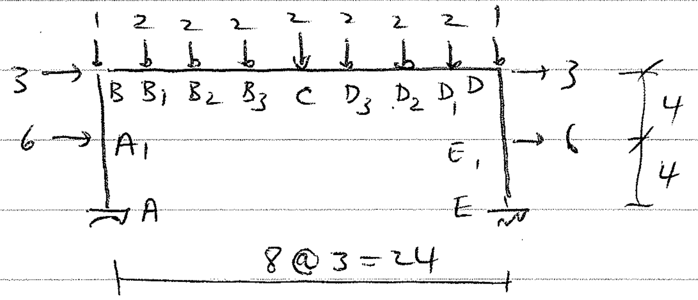
$$M_B = \tfrac{1}{4}VL - \tfrac{1}{2}Hh \qquad (3\text{-}20)$$

Thus

The V.W. EQ. may be used to determine not only the collapse equation of a frame, but also, by the consideration of various mechanisms, the complete statical analysis of the frame in that collapse state.

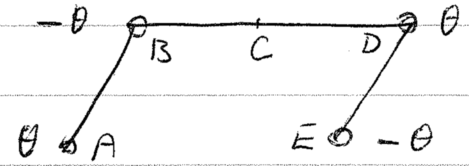
EXAMPLES

The examples of the previous chapter will now be investigated using the same sequence, but more complex loading.



	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0

First it is assumed that side sway mech. might be correct.



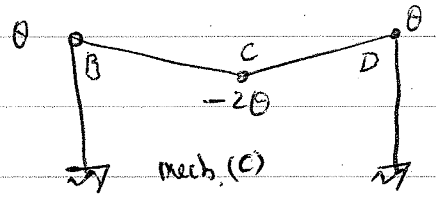
mech. (a)

Thus equation (3-11) becomes (notice all θ are cancelled)

$$(96)(1)+(48)(-1)+(48)(1)+(0)(-1)=4M_p$$

or $$M_p=24$$

The sketch of B.M. shows it is considerably in error, and as a control we calculate B.M. at C. For this purpose mechanism (c) is considered since it involves M_B, M_C and M_D and for sidsway mechanism $M_B=-24$ $M_D=24$ are known. Thus EQ (3-17) leads to



mech. (c)

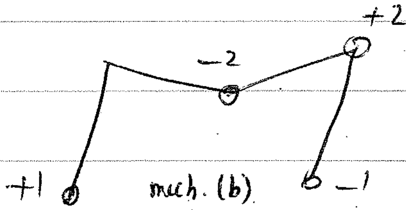
$$(-24)(1)+(M_C)(-2)+(24)(1)=(48)(1)+(0)(-2)+(48)(1)$$

or $$M_C=-48>M_p.$$

Thus second mechanism is tried, still in conjuction with Free B.M. Table (above)

$$(96)(1)+(0)(-2)+(48)(2)+(0)(-1)=6M_p$$

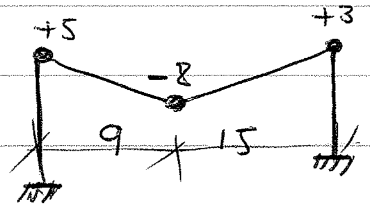
$$M_p=32$$



mech. (b)

(θ's are cancelled for simplicity)

A sketch shows that at B_3 the moment exceed M_p.
To determine M_{B_3} the mech. shown is used



However, for this M_B is needed. Therefore
first we use  and  to find M_B.
Using EQ. (3-17)

$$(M_B)(1)+(-32)(-2)+(32)(1)=(48)(1)+(0)(-2)+(48)(1)$$

or $$M_B=0 \qquad (3\text{-}24)$$

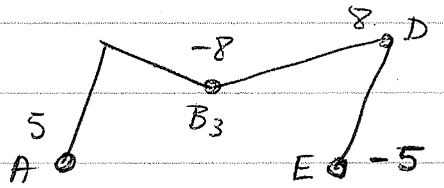
NOW we can use  to find M_{B_3}.

$$\Sigma M_i \Phi_i = \Sigma (M_F)_i \Phi_i$$

$$(0)(5)+(M_{B_3})(-8)+(32)(3)=(48)(5)+(3)(-8)+(48)(3)$$

or $$M_{B_3}=-33>-M_p$$

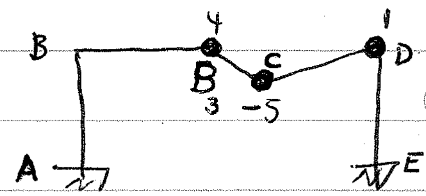
NOW that correct Mechanism is not found, we use the following:



$$(96)(5)+(3)(-8)+(48)(8)+(0)(-5)=26M_p$$

or $$M_p=32.3$$

Virtual mechanism shown is now used to determine the B.M. through out the frame. In this case M_c is found



$$(-32.3)(4)+(M_c)(-5)+(32.3)(1)=(3)(4)+(0)(-5)+(48)(1)$$

or $$M_c=-31.4 \qquad (3\text{-}27)$$

NOTE

Exactly the same method can be used for pitched roof frame.

3.3 THE FUNDAMENTAL THEOREMS

3.3.1 Assumptions: (a) proportional loading is considered, i.e each load might be thought of as having its working value and then multiplied by a common factor λ.

The fundamental Theorems are concerned with the value λ_c of the load factor at collapse of a structure.

A structure is on the point of collapse when finite deformation of at least part of the structure can occur without any change in the loads.

(b) It will be assumed that at collapse the deformations of the structure are sufficiently small to enable the effect of change of geometry on the equation of equilibrium to be neglected.

(c) It will be assumed none of the members fail by instability

It can be proved That:

THEOREM: In a structure at collapse the bending moments remain constant as the structure deforms.

COROLLARY 1: When a structure is on the point of collapse, sufficient plastic hinges must be formed to transform the structure, or part of it, into a mechanism.

COROLLARY 2: In a structure at collapse, the rate at which work is done by the external loads is equal to the work absorbed in deformation at the plastic hinges.

3.3.2 CHARACTERISTICS OF BENDING MOMENT DISTRIBUTIONS AT COLLAPSE

For any structure on the point of collapse, certain condition can be stated. The B.M. distribution must satisfy three requirements, as follows:

(1) EQUILIBRIUM CONDITION. The system of bending moments must represent a state of equilibrium between the structure and the applied loads.

(2) MECHANISM CONDITION. The bending moments must be equal to the full plastic moments of resistance at a sufficient number of sections for the structure, or part of it, to become a mechanism with plastic hinges.

(3) YIELD CONDITION. The B.M. must nowhere be allowed to exceed the full plastic moment of resistance. This is called "yield condition" because the longitudinal stress at any point in any cross-section must not exceed the yield stress.

These three conditions of collapse are a necessary requirement of a bending-moment distribution corresponding to a state of collapse, and leads to three theorems as follows.

3.3.3 UNIQUENESS THEOREM: For a structure subjected to proportional loading, if it is possible to find ~~at~~ a positive load factor λ, a bending-moment distribution satisfying the three collapse conditions, then λ is the collapse load factor λ_c and it is impossible to obtain any other positive load factor, for which a bending-moment distribution also satisfying these conditions.

PROOF: To show there is a single value for collapse load factor λ_c, let us assume it is not the case and we have two different collapse mechanisms for a particular frame under given loading, formed at different load factors λ^* and λ^{**}.

For the first of these collapse mechanisms

δ_j^* = joint displacements

θ_i^* = compatible hinge rotations

M_i^* = the collapse bending moment distribution

$\lambda^* W_j$ = external loads

and suppose yield condition is also satisfied. Therefore

(δ^*, θ^*) satisfies Mechanism condition

$(M^*, \lambda^* W)$ satisfies Equilibrium Condition

$|M^*| \leq M_p$ i.e. Yield condition

$\Big\}$ i, j are dropped for Simplicity

Using Similar Condition and employing ** we have

A: $(\lambda^* W, M^*)$ is an equilibrium set

B: $(\lambda^{**} W, M^{**})$ is an equilibrium set

C: (δ^*, θ^*) is a compatible set

D: $(\delta^{**}, \theta^{**})$ is a compatible set

For the first mechanism the collapse equation may be written by combining A and C in the Virtual work EQ.

$$\sum \lambda^* W \delta^* = \sum M^* \theta^* \qquad (3\text{-}35)$$

or

$$\lambda^* \sum W \delta^* = \sum M^* \theta^*$$

On right hand side M^* at each hinge position is $\pm M_p$. Since the plastic work dissipated in each hinge is positive, each term is + and we have

$$\lambda^* \sum W \delta^* = \sum M_p |\theta^*| \qquad (3.36)$$

Combining B and C we get

$$\sum \lambda^{**} W \delta^* = \sum M^{**} \theta^* \qquad (3.37)$$

again

$$\lambda^{**} \sum W \delta^* = \sum M^{**} \theta^*$$

on right hand side M^{**} satisfies yield condition; that is, if the two mechanism θ^* and θ^{**} have a common hinge at a certain cross section i, then M_i^{**} will be equal to $\pm M_p$, but otherwise $|M_i^{**}| < (M_p)_i$. Thus each of the terms $M_i^{**}\theta_i^*$ on the right hand side of (3.37) will be less than, or at most equal to $(M_p)_i|\theta_i^*|$, so that

$$\lambda^{**} \sum W \delta^* \leq \sum M_p |\theta^*| \qquad (3.38)$$

Comparison of (3.36) and (3-38) leads to

$$\lambda^{**} \leq \lambda^* \qquad (3\text{-}39)$$

Of the 4 statements (3.34), the last, D describing the second mechanism of collapse, has not yet been used. By combining B with D, and A with D, exactly as above,

$$\lambda^* \leq \lambda^{**} \qquad (3\text{-}40)$$

(3-39) and (3-40) are simultaneously satisfied if

$$\lambda^* = \lambda^{**} = \lambda_c$$

NOTE: It has not been established that the collapse mechanism is unique, or that B.M. distribution at collapse is unique. Indeed, it is possible that alternative mechanisms of collapse can co-exist. If they do so, however, then they must be formed at the same load factor λ_c.

3.3.4 THE UNSAFE THEOREM: If the collapse equation is written for an arbitrary assumed mechanism, then the resulting value of the load factor, λ', is always greater than, or at best equal to, the true load factor λ_c.

PROOF: The following statements will be used

$$\begin{aligned} &A: \quad (\lambda_c W, M_c) \text{ is the actual collapse distribution} \\ &B: \quad (\delta, \theta) \text{ is the assumed collapse mechanism} \end{aligned} \qquad (3.41)$$

The collapse equation written for the assumed collapse mechanism is

$$\lambda' \Sigma W \delta = \Sigma M_p |\theta| \qquad (3.42)$$

on the right-hand side, it should be noted that the value of M_p has been taken at every hinge position. Statement A and B of (3.41), when combined by equation of virtual work

$$\lambda_c \Sigma W \delta = \Sigma M_c \theta \qquad (3.43)$$

Now the values of M_c satisfy, by definition, the yield condition; that is $|M_c| \leq M_p$. Thus (3.43) leads to

$$\lambda_c \Sigma W \delta \leq \Sigma M_p |\theta| \qquad (3.44)$$

Comparing (3.42) and (3.44) leads to

$$\lambda' \geq \lambda_c$$

3.3.5 THE SAFE THEOREM: If a set of bending moments can be found at a load factor λ'' to satisfy both the equilibrium and yield conditions, then λ'' is always less than, or at most equal to, the true load factor λ_c.

PROOF: The following statements will be used

A: $(\lambda'' W, M'')$ represents a set of bending moments in equil. with the applied load, which satisfies the yield condition

B: $(\lambda_c W, M_c)$ is the actual collapse distribution (3.46)

C: (δ_c, θ_c) is the actual collapse mechanism

The value of λ_c is given by (B and C):

$$\lambda_c \sum W \delta_c = \sum M_c \theta_c = \sum M_p |\theta_c| \qquad (3.47)$$

If statements A and C of (3.46) are combined

$$\lambda'' \sum W \delta_c = \sum M'' \theta_c \qquad (3.48)$$

and since $|M''| \leq M_p$ comparing (3.47) and (3.48) leads to

$$\lambda'' \leq \lambda_c . \qquad (3.49)$$

The above three theorems can be displayed as:

$$\lambda = \lambda_c \left\{ \begin{array}{l} \text{MECHANISM CONDITION} \\ \text{EQUILIBRIUM CONDITION} \\ \text{YIELD CONDITION} \end{array} \right. \begin{array}{l} \lambda \geq \lambda_c \\ \\ \lambda \leq \lambda_c \end{array} \qquad (3\text{-}50)$$

CORROLARY 1: If material is added (of negligible self-weight) to a frame, or a restraint imposed, the frame cannot thereby be weakened. In elastic analysis this is not true, the increase of a section of a member may well increase the stress acting on that member.

PROOF: since the B.M. distribution for the collapse state of the unstrengthened frame must satisfy the yield condition, the same B.M. distribution must certainly satisfy the yield condition for the strengthened frame. The safe Theorem then states that the strengthened frame can not collapse at a load less than that of the unstrengthened frame. [M_p is increased, therefore λ'' can not be decreased] (3-42)
Similarly, the removal of material from a frame, or the removal of a constraint, can only weaken the frame; This may be seen by consideration of the unsafe Theorem. [$\Sigma M_p \theta$ will be reduced, decreasing λ'] from (3-48)

The safe Theorm is the fundamental tool of the conventional designer. For example, it is usual to consider a floor beam as simply supported and design. Now when the ends are bolted up so as to transmit moment, the beam can not be weakened.

In design if an equilibrium state is used to proportion the members so that the yield condition is satisfied, then the design is safe. This is an explanation of the fact that buildings designed in accordance with irrational codes of practice are usually safe structures.

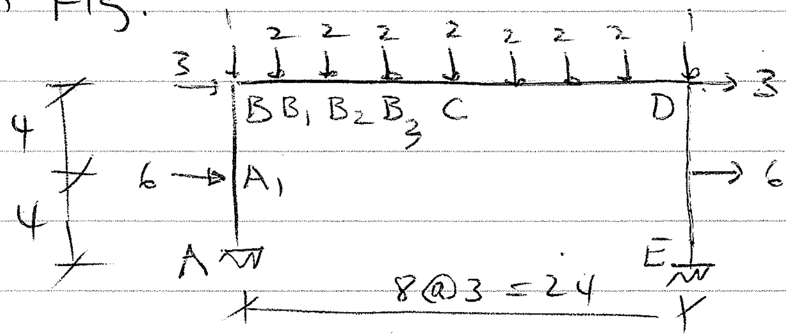
An important corollary of the unsafe Theorem is that the true load factor at collapse is the smallest possible that can be found from a consideration of all possible mechanisms of collapse. This is an important consideration in the method of combination of mechanisms discussed in the next chapter.

3.4 UPPER AND LOWER BOUNDS

The safe Theorem and The unsafe Theorem may be used together to furnish bounds on the values of the load factor λ, if the problem is one of analysis, or on the value of M_p, if the problem is one of design.

DESIGN PROBLEM

EXAMPLE 1: consider the design of a portal frame as shown in Fig.



The first trial mechanism  gave a value of $M_p = 24$. By the unsafe Theorem, the true value of M_p must be Larger Than this (or at best eqmal to 24, had the sidsway mechanism been correct, which was not in fact the case). As we saw the yield condition is violated at several sections, the worse violation occurs at B_3, where $M_{B_3} = 51$. Obviously if M_p is chosen 51, the yield condition will not violated (but the mechanism condition will not longer be satisfied). By the safe Theorem, the true value of M_p must be less Than 51. Therefore

$$24 \leq M_p \leq 51. \qquad (3.51)$$

These bounds are too wide for practical design purpose.

The second trial (mode (b))  leads to a different

bending moment distribution (Table , page)
furnishing much closer bounds

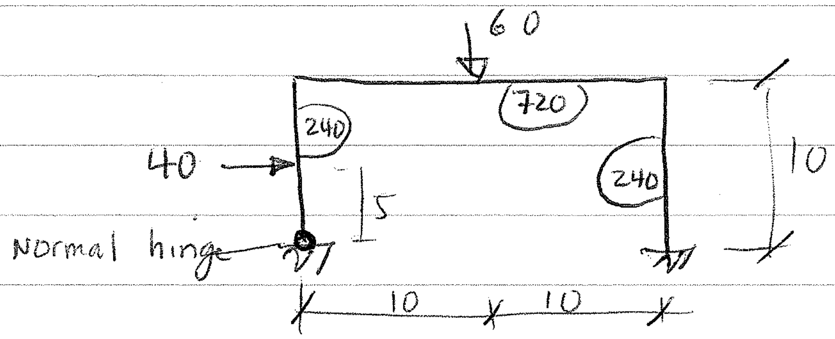
$$32 \leq M_p \leq 33 \qquad (3.52)$$

and this close enough for practical design.

Similar bounds can be established for pitched roof frame.

ANALYSIS PROBLEM

EXAMPLE 2 : We consider an example of analysis rather than design. An unsymmetrical portal frame as shown

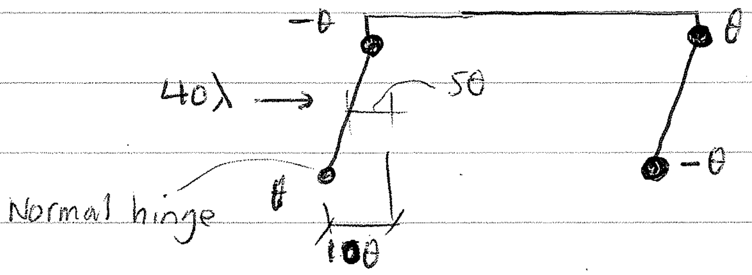


Working load is shown and the amount of λ_c is required.

SOLUTION : we start with some simple mechanisms. For example

$(40\lambda)(5\theta) = (240)(3\theta)$

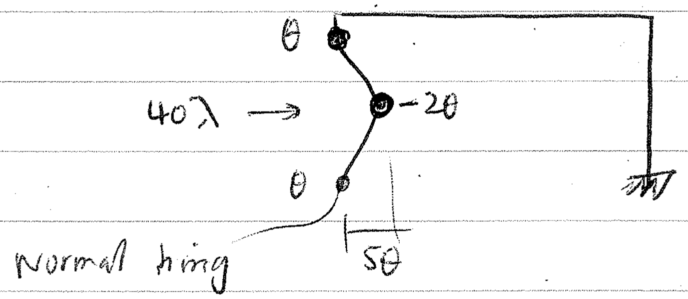
or $\lambda = 3.6$



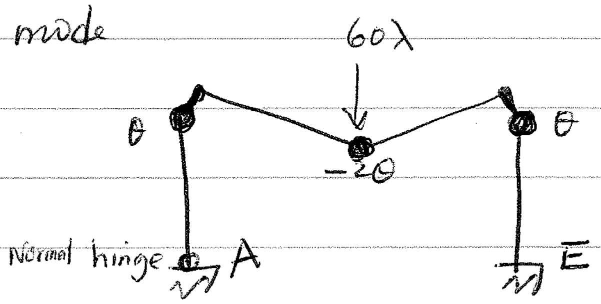
The local collapse mode

$(40\lambda)(5\theta) = (240)(3\theta)$

or $\lambda = 3.6$



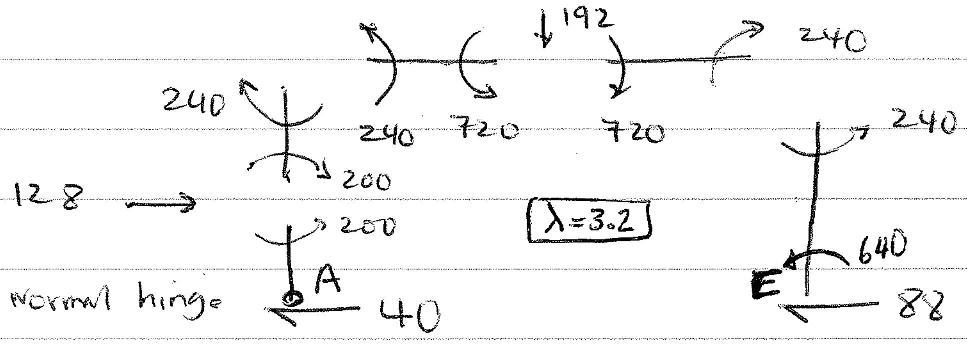
and another local collapse mode



$$(60\lambda)(10\theta) = (720)(2\theta) + (240)(2\theta)$$

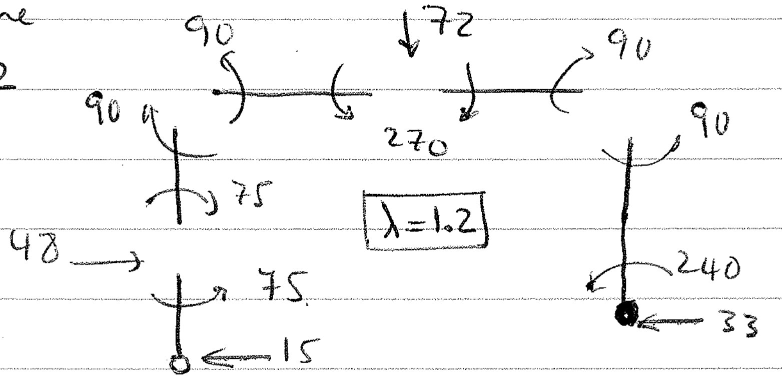
or $\lambda = 3.2$

So for $\lambda_c \leq 3.2$ with the above mode is found. Now we check the yield condition for $\lambda = 3.2$.



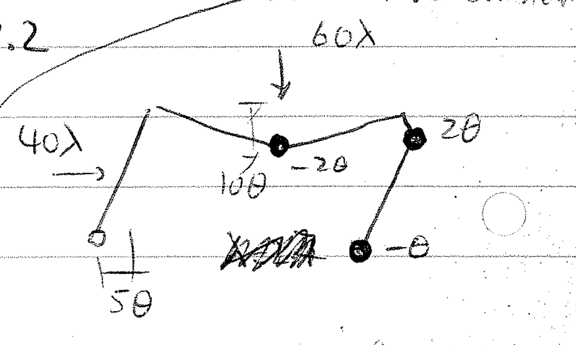
Take moment about top of left column to obtain shear 40. The other shear is $128 - 40 = 88$. with $40 \times 5 = 200$ is obtained and equil. of right column gives <u>640 at E</u>. and so on.

The yield condition is violated at E. If we multiply the above diagram by $\frac{240}{640} = \frac{3}{8}$ the following result is obtained at the load factor $\lambda = 1.2$ which just satisfies the yield condition



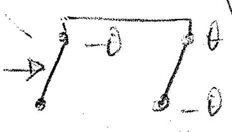
Therefore $1.2 \leq \lambda_c \leq 3.2$

~~Now let us consider~~



One can check similarly for sway mechanism, leading to

$$1.8 \leq \lambda_c \leq 3.6$$

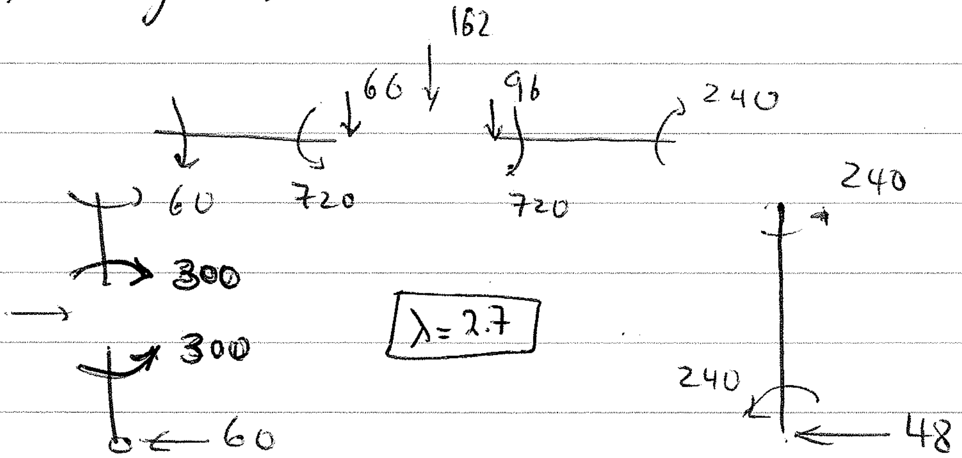


The bending moment indicates that this mechanism may give better result (for collapse load factor).
The collapse equation is

$$(40\lambda)(5\theta)+(60\lambda)(10\theta)=(720)(2\theta)+(240)(3\theta)$$

or $$\lambda = 2.7 \qquad (3.57)$$

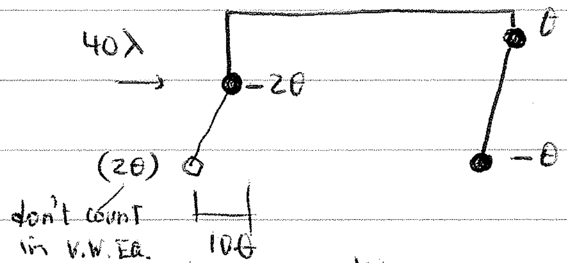
The statical analysis leads to



Yield condition is violated in left-hand column. A safe solution is obtained by multiplying the above diagram at $\frac{240}{300} = 0.8$ so that

$$2.16 \leq \lambda_c \leq 2.7$$

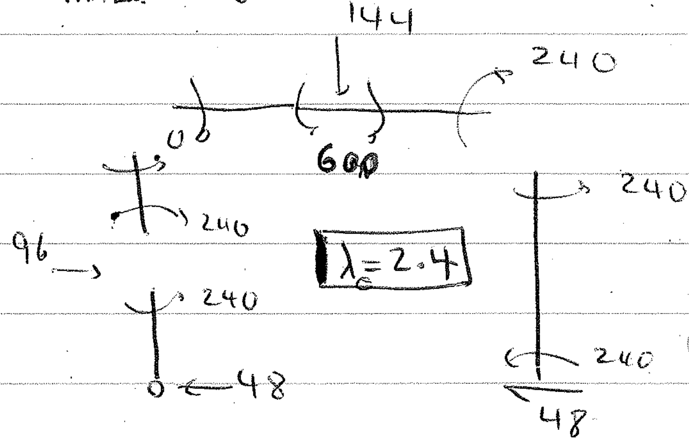
The correct mechanism of collapse is shown →. collapse equation is



$$(40\lambda)(10\theta)=(240)(4\theta)$$

or $\lambda = 2.4$

statical analysis is displayed



You can see that trial-error method is very lengthy. Other methods are given in next chapter.

Yield is satisfied.